The Ehrhart polynomial of an integral zonotope

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The space D of all polynomials in the linear span S of the integer translates of a box spline is of basic importance in multivariate spline theory, since it allows the construction of good approximation maps to S from spaces containing S. Being closed under differentiation, D is the joint kernel of a set I_D of differential operators with constant coefficients. Even though I_D is known explicitly, D is hard to construct. At the same time, a polynomial space P is known that serves as a natural dual of D. The duality between D and P is used to construct linear projectors onto a box spline space, to solve interpolation problems induced by P, and to determine the local approximation order of some exponential spaces.

We consider two natural companions of P, the polynomial spaces P_- and P_+, which form the chain

$$P_- \subset P \subset P_+.$$

The structure of the spaces P_-, P, and P_+ is closely related to the geometry of the zonotope Z that is the support of the box spline. In particular, if the direction set of Z is integral, finding the dimensions of P_-, P, and P_+ requires counting the number of integer point in the interior Z°, in the half-open half-closed variant $Z^\cdot$ of Z, and in all of Z, respectively. This further leads to the main question answered here

Q: Find a formula for the Ehrhart polynomial E_Z associated with Z.

We show that the k-th coefficient of E_Z is the sum of the volumes of all linearly independent subsets of cardinality k of the direction set of Z.

The result suggests the existence of a complete covering of Z by j-th dimensional half-open half-closed parallelepipeds each of which corresponds to exactly one of the linearly independent subsets of the direction set of Z for any (not necessarily integral) zonotope Z.