Wavelet Frames: The Power of Redundant MultiResolution Representation

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ftp site: `ftp://ftp.cs.wisc.edu/Approx`

Outline

(1) Multiscale representations

(2) Redundant multiscale representations

(3) Applications: examples of compression and denoising

(4) Frames introduced

(5) A glimpse into the history and theory of frames and wavelet frames

(6) Extension principles. Tight spline frames

(7) The fast frame transform. The IDR **FrameNet** project.

(8) The promise and challenges in exploiting redundancy

(9) Applications revisited

(1): **Multiscale representations in Euclidean domains**

Goal. *space-frequency representation of functions: a transform*

$$Tf : \mathbb{R}^d \times \mathbb{R}^d,$$

such that

$$Tf(x, \omega)$$

is the "content" of f at space location x and frequency location ω.

Linear T, discrete version. *X is a countable collection (=* **system***) of functions (linear functionals), T consists of* **analysis/decomposition***:*

$$T : f \mapsto \{\langle f, x\rangle : \ x \in X\}.$$

Weyl-Heisenberg systems. *Complete symmetry between space and frequency. Both domains are translation-invariant. Corresponds to Wiener's amalgam spaces.*

Wavelet systems. *We must allow "zoom in" capabilities in space, since the function* f *may record spatial transient events. Corresponds to Sobolev and Besov spaces.*

Synthesis. *We associate the system* X *with a* **dual system** $\widetilde{X} = \{\widetilde{x} : \; x \in X\}$. *We then recover* f *by inverting* T*:*

$$f = \sum_{x \in X} \langle f, x \rangle \widetilde{x}.$$

Example. X *is a complete orthonormal basis.* $\widetilde{X} = X$.

(2): Redundant multiscale representations

Reminder: Synthesis. *We associate the system* X *with a* **dual system** $\widetilde{X} = \{\widetilde{x} : x \in X\}$. *We then recover* f *by inverting* T*:*

- $$f = \sum_{x \in X} \langle f, x \rangle \widetilde{x}.$$

There are several ways to insert **redundancy**.

(1) Utilize simultaneously several related transforms. Select a transform that best represents f.

Example: **wavelet packets**. Goes also under the name of **best basis pursuit**

(2) Employ a system X and a dual system $\widetilde{X}$ which are **dependent**. Then, there are many alternative representations to •. Choose an optimal one.

Example: **frames**. Goes by names like **matching pursuit, greedy algorithms, principal component analysis...**

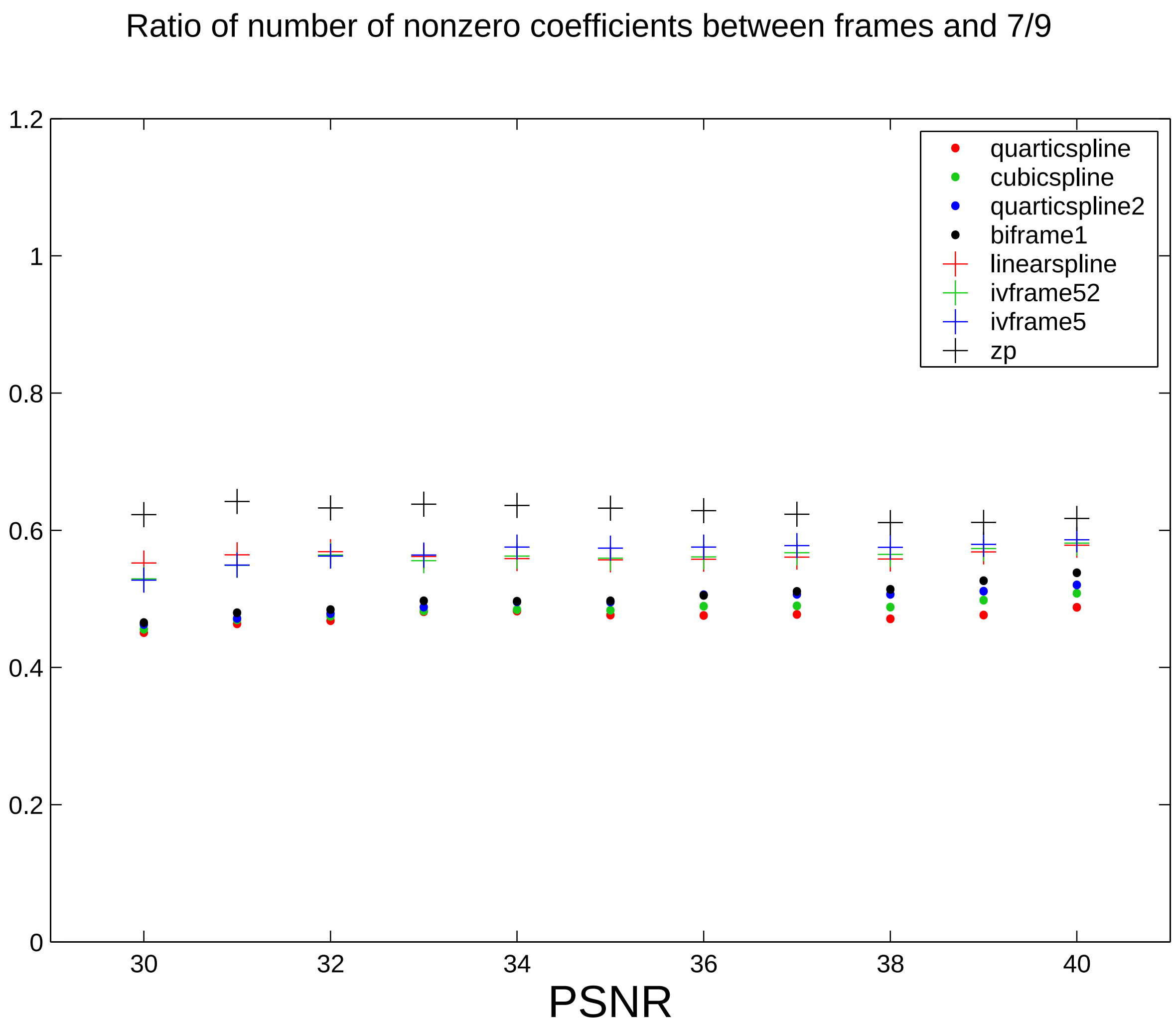
(3): Examples of Applications

(1) Image Compression

(2) Feature Detection and denoising

Let's begin with the latter...

Goldhill: A (standard) 128×128 b&w test image



A comparison: of sparsity between the wavelet 7/9 and several framelets

(4) Frames introduced

Definition: **Frames.** *$X \subset L_2(\mathbb{R}^d)$ is countable.*

X is a frame if, for some $C_1, C_2 > 0$,

$$C_1\|f\|^2 \leq \sum_{x\in X} |\langle f, x\rangle|^2 \leq C_2\|f\|^2, \ \forall f \in L_2(\mathbb{R}^d).$$

Definition. **tight frames***:*

the above holds for $C_1 = C_2 = 1$.

Discussion. *The definition of frames allows* **redundancy***.*

Definition. **A wavelet system***:*

$\boldsymbol{\Psi} \subset L_2(\mathbb{R}^d)$ is finite. The wavelet system $X(\boldsymbol{\Psi})$ is then

$$2^{kd/2}\boldsymbol{\psi}(2^k \cdot +j), \quad \boldsymbol{\psi} \in \boldsymbol{\Psi}, \ k \in \mathbb{Z}, \ j \in \mathbb{Z}^d.$$

(5) A glimpse into the history and theory of frames and wavelet frames

History

1950's:

frames are introduced by R.J. Duffin and A.C. Schaeffer

1986:

I. Daubechies, A. Grossmann, and Y. Meyer construct 1D tight wavelet frames.

1980's & 1990's:

J. Benedetto, his students (C. Heil, D, Walnut, W. Heller...), and others (H. Feichtinger, P. Casazza, G. Gröchenig) study wavelet and Weyl-Heisenberg frames.

1991-1992:

I. Daubechies popularizes frames
(in her book, and in her IEEE survey)

1990's:

Studies of wavelet frames by several groups: Weiss, Larson, Chui-Shi (oversampling)...

1994-1995:

the duality principle of Weyl-Heisenberg frames is found

(independently by:

(1)I.J.E.M. Janssen,

(2)I. Daubechies-H. Landau-Z. Landau,

(3)A.R - Z. Shen)

1995-1996:

a cohesive theory of wavelet frames is established

Since then: a wealth of possible constructions. Fast implementation. A few experiments with applications.

Major challenges: understanding the performance of the basic algorithms in the context of redundancy.

The theory of wavelet frames

was derived with the aid of the theory of shift-invariant spaces

(de Boor-DeVore-R, JFA, 1994; R-Shen, CJM, 1995)

Hallmarks

(R-Shen, JFA, JFAA, 1997)

(1) The introduction of quasi-affine systems:

A link between wavelets and shift-invariant space theory.

(2) Fiberization of wavelet systems:

A study of wavelet systems via a related collection of constant coefficient matrices (= the fibers).

(3) A complete characterization of all wavelet frames,

in terms of the invertability of the fiber operators.

(4) An induced characterization of tight frames and bi-frames.

(5) framelets: a complete characterization of all MRA-based wavelet frames

(6) framelets: the unitary extension principle (UEP)

Framelets: MRA construction of wavelet frames (R-Shen)

The MRA setup:

(1) $V_0 \subset L_2(\mathbb{R}^d)$ is PSI, i.e., it is spanned by $\boldsymbol{\phi}(\cdot + \alpha)$, $\alpha \in \mathbb{Z}^d$, for some function $\boldsymbol{\phi}$.

(2) $V_1 := V_0(2\cdot)$ is a superspace of V_0.

(3) The **mother wavelets** $\boldsymbol{\Psi}$ are selected from V_1.

Q.: under what conditions the system $X(\boldsymbol{\Psi})$ is a frame?

Since $\boldsymbol{\phi}, \boldsymbol{\Psi} \subset V_1$, $\exists$ 2π-periodic functions $\boldsymbol{\tau_\phi}$, $\boldsymbol{\tau_\psi}$, $\boldsymbol{\psi} \in \boldsymbol{\Psi}$, s.t.

$$\widehat{\boldsymbol{\phi}}(2\cdot) = \boldsymbol{\tau_\phi}\widehat{\boldsymbol{\phi}}, \quad \widehat{\boldsymbol{\psi}}(2\cdot) = \boldsymbol{\tau_\psi}\widehat{\boldsymbol{\phi}}.$$

Definition. **the fundamental function of the MRA construction.**

$$\Theta(\omega) := \sum_{k=0}^{\infty} \prod_{m=0}^{k-1} |\boldsymbol{\tau_\phi}(2^m\omega)|^2 \sum_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} |\boldsymbol{\tau_\psi}(2^k\omega)|^2.$$

Theorem. (R-Shen, 97) $\widehat{\boldsymbol{\phi}}(0) = 1$. *$X(\boldsymbol{\Psi})$ is a tight frame if and only if:*

(i) $\Theta(0) = 1$.

(ii) For every $\nu \in \{0, \pi\}^d \backslash 0$

$$\Theta(2\omega)\boldsymbol{\tau_\phi}(\omega)\overline{\boldsymbol{\tau_\phi}(\omega+\nu)} + \sum_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} \boldsymbol{\tau_\psi}(\omega)\overline{\boldsymbol{\tau_\psi}(\omega+\nu)} = 0.$$

Q. How to use this theorem for actual constructions?

Theorem: the Unitary Extension Principle. (R-Shen, JFA, 97)

Assume that for every $\nu \in \{0, \pi\}^d$

$$\boldsymbol{\tau}_{\boldsymbol{\phi}}(\omega)\overline{\boldsymbol{\tau}_{\boldsymbol{\phi}}(\omega+\nu)} + \sum_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} \boldsymbol{\tau}_{\boldsymbol{\psi}}(\omega)\overline{\boldsymbol{\tau}_{\boldsymbol{\psi}}(\omega+\nu)} = \delta_\nu.$$

Then $X(\boldsymbol{\Psi})$ *is a tight frame.*

Theorem: the Oblique Extension Principle. (Daubechies-Han-R-Shen, 01)

$X(\boldsymbol{\Psi})$ *is a tight framelet iff there exists a non-negative* Θ *such that:*

(a) $\Theta(0) = 1$.

(b) For every $\nu \in \{0, \pi\}^d$

$$\Theta(2\omega)\boldsymbol{\tau}_{\boldsymbol{\phi}}(\omega)\overline{\boldsymbol{\tau}_{\boldsymbol{\phi}}(\omega+\nu)} + \sum_{\boldsymbol{\psi}\in\boldsymbol{\Psi}} \boldsymbol{\tau}_{\boldsymbol{\psi}}(\omega)\overline{\boldsymbol{\tau}_{\boldsymbol{\psi}}(\omega+\nu)} = \delta_\nu\Theta.$$

The punch line: It is easier to construct wavelet frames compared to orthonoromal/biorthogonal wavelet systems!

So, we have useful construction methods...

what guidelines should we follow?

(1) Short filters

(2) Smooth wavelets

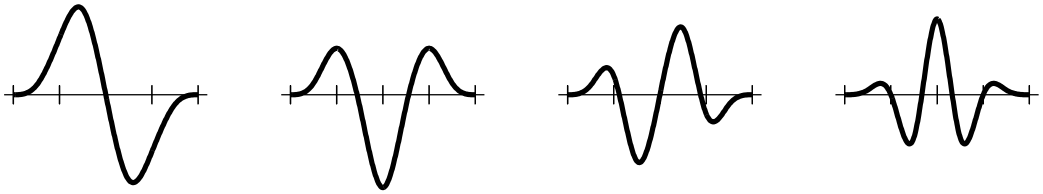
(3) Many different windows?

(4) High approximation orders

(5) Low/high oversampling rate

(6) High/varying vanishing moments

(7) Artifact freeness (splines)



RS4: A cubic spline 5-tap tight frame

Spline tight framelets (R-Shen, 97):

(1) Shortest filters (5-tap for cubic splines)

(2) All possible vanishing moments (1..m for order m)

(3) large oversampling rates (m for order m)

(4) pure averages/differences

(5) Low approximation order (2)

(7) The fast frame transform. The IDR FrameNet project.

(8) The promise and challenges in exploiting redundancy

The Promise

(1) More flexibility in building representation systems.

(2) An ability to search for an optimal representation among a well-defined class.

The Challenge

(1) Obtaining rigorous understanding of the performance of standard algorithms (thresholding, quantization) for redundant representations

(2) Quantifying the gain in redundancy.

(3) Developing low-complexity algorithms for finding an optimal representation.

(9) Applications revisited